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Optimum Low-Thrust Rendezvous and Station Keeping

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Analytic solutions are determined for the optimum correction of all six elements of elliptic satellite orbits with low-thrust, power-limited propulsion systems. The optimum direction and magnitude of thrust are determined as functions of time so as to minimize the fuel required to rendezvous in a given time or to minimize the fuel required for station keeping in the presence of known perturbations. The motion of the vehicle under the action of the optimum thrust program is also determined analytically. The solutions obtained are divided into two classes, depending on whether rendezvous requires few or many revolutions. In the latter case, simple analytic solutions are obtained explicitly by neglecting short-period perturbations.

Nomenclature

a	= semimajor axis
A	= thrust acceleration magnitude
A_T	= circumferential component of thrust acceleration
A_R	= radial component of thrust acceleration
A_W	= normal component of thrust acceleration
e	= eccentricity
E	= eccentric anomaly
F	= variational Hamiltonian
J	= defined by Eq. (1)
K	= gravitational constant
$L, M, \ldots, U$	= functions defined following Eq. (7)
M	= mean anomaly
t	= time
λ	= Lagrange multiplier
θ_1	= small rotation of major axis in orbit plane
θ_2	= small rotation of orbit plane around major axis
θ_3	= small rotation of orbit plane around latus rectum

Introduction

THE present paper is one of a series of papers[1–4] containing analytic solutions for optimum low-thrust power-limited trajectories in inverse-square force fields. The previous papers in this series have considered long-time transfer between arbitrary circular orbits,[1] long-time escape from elliptic orbits,[2] long-time transfer between coplanar circular and elliptic orbits,[3] and transfer between close circular orbits.[4]

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The propulsion system that effects the changes of the elliptic orbit is assumed to be power-limited, i.e., it operates at constant exhaust power with a thrust magnitude inversely proportional to the exhaust velocity.[5,6] The direction and magnitude of the thrust, which is assumed to be completely variable, is to be determined as a function of time so as to minimize fuel consumption.

There have been a number of previous studies of trajectory optimization for power-limited propulsion systems; some of the more important studies are contained in Refs. 5–13. Most of these studies are concerned with numerical results for inverse-square force fields.[5–11] Reference 12 obtains analytic solutions for optimum trajectories in field-free space, whereas Ref. 13 obtains analytic solutions for the optimum thrust programs for small changes of five elements of an elliptic orbit in an inverse-square field. The present paper extends the work of Ref. 13 to obtain the optimum thrust program for small changes of all six elements and to determine the optimal motion of the vehicle and the fuel consumption under the action of this thrust program.

Assumptions

The analyses of this paper are carried out under the assumption that the perturbations due to thrust of all the elements of an elliptic orbit are small and that the thrust acceleration A is small compared to the acceleration of gravity. The other assumptions are an inverse-square force field and a variable-thrust powerplant operating at constant exhaust power with no bounds on the thrust magnitude or direction.

Derivation

The derivation is carried out using a formulation of the problem of Bolza similar to that of Breakwell[14] and of Pontryagin.[15] The payoff to be maximized is the negative of the integral J of Ref. 5 (J is to be minimized). It is shown in Ref. 5 that J is a measure of the fuel consumption. The six state variables (a, e, θ_1, θ_2, θ_3, M) are six parameters that completely specify an elliptic orbit. Since only small changes in this orbit are to be considered, an approximate set of orbital parameters, similar to those of Stromgren[16] has been used. The orbital elements θ_1, θ_2, and θ_3 represent three orthogonal rotations. θ_1 represents a small rotation of the line of apsides in the plane of the orbit, θ_2 represents a small rotation of the orbit plane about the major axis, and θ_3 represents a small rotation of the orbit plane about the latus-rectum. The orbital elements a, e, E, and M are semimajor axis, eccentricity, eccentric anomaly, and mean anomaly, respectively. The control variables A_T, A_R, and A_W are three orthogonal components of the thrust acceleration A, A_T being circumferential, A_R being radial, and A_W being normal to the orbit plane. The independent variable is time. Equation (1) is the equation for the rate of change of the payoff, and Eqs. (2) to (7) are the equations of motion:

$$-\frac{dJ}{dt} = f_0 = -\frac{A_T^2 + A_R^2 + A_W^2}{2} \tag{1}$$

$$\dot{x}_1 = da/dt = f_1 = A_T L + A_R M \tag{2}$$

$$\dot{x}_2 = de/dt = f_2 = A_T N + A_R O \tag{3}$$

$$\dot{x}_3 = d\theta_1/dt = f_3 = A_T P + A_R Q \tag{4}$$

$$\dot{x}_4 = d\theta_2/dt = f_4 = A_W R \tag{5}$$

$$\dot{x}_5 = d\theta_3/dt = f_5 = A_W S \tag{6}$$

$$\dot{x}_6 = dM/dt = f_6 = A_T T + A_R U + (K/a^3)^{1/2} \tag{7}$$

$$L = 2\left(\frac{a^3}{K}\right)^{1/2} \frac{(1-e^2)^{1/2}}{1 - e\cos E}$$

$$M = 2\left(\frac{a^3}{K}\right)^{1/2} \frac{e\sin E}{1 - e\cos E}$$

$$N = \left[\frac{a}{K}(1-e^2)\right]^{1/2} \frac{2\cos E - e\cos^2 E - e}{1 - e\cos E}$$

$$O = \left(\frac{a}{K}\right)^{1/2} \frac{(1-e^2)\sin E}{1 - e\cos E}$$

$$P = \left(\frac{a}{K}\right)^{1/2} \frac{2 - e\cos E - e^2}{e(1 - e\cos E)} \sin E$$

$$Q = -\left[\frac{a}{K}(1-e^2)\right]^{1/2} \frac{\cos E - e}{e(1 - e\cos E)}$$

$$R = \left(\frac{a}{K}\right)^{1/2} \frac{\cos E - e}{(1-e^2)^{1/2}}$$

$$S = (a/K)^{1/2} \sin E$$

$$T = -(1-e^2)^{1/2} P$$

$$U = -(1-e^2)^{1/2} Q - 2(a/K)^{1/2}(1 - e\cos E)$$

The variables M and E are related by Kepler's equation:

$$M = E - e\sin E \tag{8}$$

Since only small perturbations of an elliptic orbit are to be considered, the eccentric anomaly may be regarded as the eccentric anomaly of the unperturbed orbit. The difference between the perturbed and unperturbed eccentric anomaly will be of order A and will introduce terms of order A^2 into the integrated equations for the perturbations. These terms may be neglected, and

$$E - e^0 \sin E = [K/(a^0)^3]^{1/2} t + E^0 - e^0 \sin E^0 \tag{9}$$

The superscript 0 refers to initial values of the variables. The variational Hamiltonian is defined by Eq. (10), where the λ_i are variable Lagrange multipliers:

$$F = f_0 + \sum_{i=1}^{6} \lambda_i f_i \tag{10}$$

The Euler-Lagrange equations for the three control variables are given by Eqs. (11) and (12):

$$\frac{\partial F}{\partial A_T} = 0 \qquad \frac{\partial F}{\partial A_R} = 0 \qquad \frac{\partial F}{\partial A_W} = 0 \tag{11}$$

$$\begin{aligned} A_T &= \lambda_1 L + \lambda_2 N + \lambda_3 P + \lambda_6 T \\ A_R &= \lambda_1 M + \lambda_2 O + \lambda_3 Q + \lambda_6 U \\ A_W &= \lambda_4 R + \lambda_5 S \end{aligned} \tag{12}$$

The Legendre condition for F to be a maximum is that the second partials of F, with respect to the control variables, be negative definite at the stationary point. These conditions are fulfilled by Eqs. (13):

$$\partial^2 F/\partial A_T^2 = -1 < 0$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial A_T^2} & \dfrac{\partial^2 F}{\partial A_T \partial A_R} \\ \dfrac{\partial^2 F}{\partial A_R \partial A_T} & \dfrac{\partial^2 F}{\partial A_R^2} \end{vmatrix} = 1 > 0 \qquad \begin{vmatrix} \dfrac{\partial^2 F}{\partial A_T^2} & \dfrac{\partial^2 F}{\partial A_T \partial A_R} & \dfrac{\partial^2 F}{\partial A_T \partial A_W} \\ \dfrac{\partial^2 F}{\partial A_R \partial A_T} & \dfrac{\partial^2 F}{\partial A_R^2} & \dfrac{\partial^2 F}{\partial A_R \partial A_W} \\ \dfrac{\partial^2 F}{\partial A_W \partial A_T} & \dfrac{\partial^2 F}{\partial A_W \partial A_R} & \dfrac{\partial^2 F}{\partial A_W^2} \end{vmatrix} = -1 < 0 \tag{13}$$

Since Eqs. (13) hold over the entire field of admissible controls as well as at the stationary point, the stationary point given by Eqs. (12) is an absolute maximum, and the Weierstrass condition (or equivalently, the Pontryagin maximum principle) is satisfied.

The remaining Euler-Lagrange equations specify the rates of change of the Lagrange multipliers:

$$d\lambda_i/dt = -(\partial F/\partial x_i) \qquad i = 1, 2, \ldots, 6 \tag{14}$$

These six equations are obtained from Eqs. (1–7, 10, and 12) and the assumption, consistent with Eq. (12), that $\lambda_i = O(A)$

$$\frac{d\lambda_1}{dt} = \frac{3}{2}\lambda_6 \left[\frac{K}{(a^0)^5}\right]^{1/2} + O(A^2) \tag{15}$$

$$\frac{d\lambda_i}{dt} = O(A^2) \qquad i = 2, 3, 4, 5, 6 \tag{16}$$

Integrating Eqs. (15) and (16),

$$\lambda_1 = \lambda_1^0 + \frac{3}{2}\lambda_6 \left[\frac{K}{(a^0)^5}\right]^{1/2} t + O(A^2) \tag{17}$$

$$\lambda_i = \lambda_i^0 + O(A^2) \qquad i = 2, 3, 4, 5, 6 \tag{18}$$

Substituting Eqs. (17) and (18) into Eqs. (12), the optimum thrust programs are given explicitly by Eqs. (19):

$$\left.\begin{aligned} A_T &= \lambda_1^0 L + \lambda_2^0 N + \lambda_3^0 P + \\ &\qquad \lambda_6^0 \left(T + \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2} tL\right) + O(A^2) \\ A_R &= \lambda_1^0 M + \lambda_2^0 O + \lambda_3^0 Q + \\ &\qquad \lambda_6^0 \left(U + \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2} tM\right) + O(A^2) \\ A_W &= \lambda_4^0 R + \lambda_5^0 S + O(A^2) \end{aligned}\right\} \tag{19}$$

These equations represent the first derivation of the optimum power-limited thrust program for effecting small changes in all six elements of an elliptic orbit. Ross and Leitmann[13] have previously derived the optimum program for changing five elements. Their results for these elements, derived by an entirely different procedure, agree with those presented herein. Lawden[17] has obtained results completely equivalent to those presented here for coasting arcs with constant-exhaust-velocity propulsion. By changing the payoff function in Ref. 17 to the one considered herein, his results may be transformed to those obtained herein for all six elements.

Equations (19) may now be substituted into Eqs. (1–7) and the results integrated, using unperturbed elements and neglecting terms of order A^2. For these integrations, the independent variable is changed to eccentric anomaly. From Eq. (9),

$$dt = [(a^0)^3/K]^{1/2}(1 - e^0 \cos E)dE \tag{20}$$

The correct first-order terms for the rates of change of all elements except the mean anomaly may be found by simply substituting Eqs. (19) into Eqs. (1–7) using Eqs. (9) and (20). The first-order perturbations in semimajor axis produce first-order perturbations in mean anomaly so that Eq. (21) must be used to calculate the mean anomaly to first order:

$$\begin{aligned}\frac{dM}{dt} &= A_T T + A_R U + \left[\frac{K}{(a^0)^3}\right]^{1/2} - \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2}\Delta a \\ &= A_T T + A_R U + \left[\frac{K}{(a^0)^3}\right]^{1/2} - \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2}\frac{d(t\Delta a)}{dt} + \\ &\qquad \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2} t\frac{da}{dt}\end{aligned} \tag{21}$$

The second form of Eq. (21) will be used in the subsequent development.

Two more definitions [Eq. (22)] will be used before writing out the first-order rates of change of all the elements. They are

$$\begin{aligned}T + \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2} tL &\equiv V \\ U + \frac{3}{2}\left[\frac{K}{(a^0)^5}\right]^{1/2} tM &\equiv W\end{aligned} \tag{22}$$

With these definitions, the rates of change of all elements are given by Eqs. (23–29). In these equations and the subsequent ones, the 0 superscript for initial values of the elements and of λ_i has been omitted but should be understood:

$$\frac{da}{dt} = \lambda_1(L^2 + M^2) + \lambda_2(LN + MO) + \lambda_3(LP + MQ) + \lambda_6(LV + MW) \tag{23}$$

$$\frac{de}{dt} = \lambda_1(LN + MO) + \lambda_2(N^2 + O^2) + \lambda_3(NP + OQ) + \lambda_6(NV + OW) \tag{24}$$

$$\frac{d\theta_1}{dt} = \lambda_1(LP + MQ) + \lambda_2(NP + OQ) + \lambda_3(P^2 + Q^2) + \lambda_6(PV + QW) \tag{25}$$

$$\frac{d\theta_2}{dt} = \lambda_4 R^2 + \lambda_5 RS \tag{26}$$

$$\frac{d\theta_3}{dt} = \lambda_4 RS + \lambda_5 S^2 \tag{27}$$

$$\frac{dM}{dt} = \left(\frac{K}{a^3}\right)^{1/2} + \lambda_1(LV + MW) + \lambda_2(NV + OW) + \lambda_3(PV + QW) + \lambda_6(V^2 + W^2) - \frac{3}{2}\left(\frac{K}{a^5}\right)^{1/2}\frac{d(t\Delta a)}{dt} \tag{28}$$

$$\frac{dJ}{dt} = \frac{1}{2}\sum_{i=1}^{6}\lambda_i\frac{dx_i}{dt} - \frac{\lambda_6}{2}\left(\frac{K}{a^3}\right)^{1/2} + \frac{3}{4}\left(\frac{K}{a^5}\right)^{1/2}\frac{d(t\Delta a)}{a^3} \tag{29}$$

Carrying out these integrations with eccentric anomaly as independent variable results in Eqs. (30–37). The limits for these integrations are taken from the initial value of eccentric anomaly to the current value:

$$\Delta a = \lambda_1\phi_{11} + \lambda_2\phi_{12} + \lambda_3\phi_{13} + \lambda_6\phi_{16} \tag{30}$$

$$\Delta e = \lambda_1\phi_{12} + \lambda_2\phi_{22} + \lambda_3\phi_{23} + \lambda_6\phi_{26} \tag{31}$$

$$\Delta\theta_1 = \lambda_1\phi_{13} + \lambda_2\phi_{23} + \lambda_3\phi_{33} + \lambda_6\phi_{36} \tag{32}$$

$$\Delta\theta_2 = \lambda_4\phi_{44} + \lambda_5\phi_{45} \tag{33}$$

$$\Delta\theta_3 = \lambda_4\phi_{45} + \lambda_5\phi_{55} \tag{34}$$

$$M = (K/a^3)^{1/2}t + \lambda_1\phi_{16} + \lambda_2\phi_{26} + \lambda_3\phi_{36} + \lambda_6\phi_{66} - (\tfrac{3}{2})(K/a^5)^{1/2}t\Delta a \tag{35}$$

$$J = \tfrac{1}{2}(\lambda_1\Delta a + \lambda_2\Delta e + \lambda_3\Delta\theta_1 + \lambda_4\Delta\theta_2 + \lambda_5\Delta\theta_3 + \lambda_6\lambda_1\phi_{16} + \lambda_6\lambda_2\phi_{26} + \lambda_6\lambda_3\phi_{36} + \lambda_6^2\phi_{66}) \tag{36}$$

$$t = (a^3/K)^{1/2}(E - e\sin E - E^0 + e\sin E^0) = (a^3/K)^{1/2}(E - e\sin E - M^0) \tag{37}$$

$$\phi_{11} = (a^9/K^3)^{1/2}4[E + e\sin E]_{E^0}^{E}$$

$$\phi_{12} = (a^7/K^3)^{1/2}4(1 - e^2)[\sin E]_{E^0}^{E}$$

$$\phi_{13} = -\left(\frac{a^7}{K^3}\right)^{1/2} 4\,\frac{(1 - e^2)^{1/2}}{e}[\cos E]_{E^0}^{E}$$

$$\phi_{16} = \left(\frac{a^7}{K^3}\right)^{1/2}\left[3E^2 + 6eE\sin E + 4\,\frac{1 + 3e^2}{e}\cos E + e^2\cos^2 E - 6M^0(E + e\sin E)\right]_{E^0}^{E}$$

$$\phi_{22} = \left(\frac{a^5}{K^3}\right)^{1/2}(1 - e^2)\left[\frac{5}{2}E - 4e\sin E + \frac{3}{2}\sin E\cos E + \frac{e}{3}\sin^3 E\right]_{E^0}^{E}$$

$$\phi_{23} = \left(\frac{a^5}{K^3}\right)^{1/2}(1 - e^2)^{1/2}\left[\cos E - \frac{3 - e^2}{2e}\cos^2 E + \frac{1}{3}\cos^3 E\right]_{E^0}^{E}$$

$$\phi_{26} = \left(\frac{a^5}{K^3}\right)^{1/2}(1 - e^2)\left[6E\sin E + 7\cos E + \frac{3(1 + e^2)}{2e}\cos^2 E - \frac{\cos^3 E}{3} - 6M^0\sin E\right]_{E^0}^{E}$$

$$\phi_{33} = \left(\frac{a^5}{K^3}\right)^{1/2}\left[\frac{5 - 4e^2}{2e^2}E - \frac{1 - e^2}{e}\sin E - \frac{3 - 2e^2}{2e^2}\sin E\cos E - \frac{\sin^3 E}{3e}\right]_{E^0}^{E}$$

$$\phi_{36} = \left(\frac{a^5}{K^3}\right)^{1/2}(1 - e^2)^{1/2}\left[-\frac{5 + 8e^2}{2e^2}E - \frac{6E\cos E}{e} + \frac{9 + e^2}{e}\sin E + \frac{3 + 2e^2}{2e^2}\sin E\cos E + \frac{\sin^3 E}{3e} + \frac{6}{e}M^0\cos E\right]_{E^0}^{E}$$

$$\phi_{44} = \left(\frac{a^5}{K^3}\right)^{1/2} \frac{1}{1-e^2} \left[\frac{1+4e^2}{2} E - e(3+e^2)\sin E + \frac{1+e^2}{2} \sin E \cos E + \frac{e}{3} \sin^3 E\right]_{E^0}^{E}$$

$$\phi_{45} = \left(\frac{a^5}{K^3}\right)^{1/2} \frac{1}{(1-e^2)^{1/2}} \left[e \cos E - \frac{1+e^2}{2} \cos^2 E + \frac{e}{3} \cos^3 E\right]_{E^0}^{E}$$

$$\phi_{55} = \left(\frac{a^5}{K^3}\right)^{1/2} \left[\frac{E}{2} - \frac{\sin E \cos E}{2} - \frac{e}{3} \sin^3 E\right]_{E^0}^{E}$$

$$\phi_{66} = \left(\frac{a^5}{K^3}\right)^{1/2} \left[3E^3 + \frac{5+23e^2+13e^4}{2e^2} E + 9eE^2 \sin E + \frac{3}{2} e^2 E(\cos^2 E - \sin^2 E) + 12 \frac{1+3e^2}{e} E \cos E - \frac{17+46e^2+e^4}{e} \sin E - 3 \frac{1+e^2+2e^4}{2e^2} \sin E \cos E - \frac{1-e^2-e^4}{3e} \sin^3 E - 3M^0 \left(3E^2 + 6eE \sin E + 4 \frac{1+3e^2}{e} \cos E + e^2 \cos^2 E\right) + 9(M^0)^2 (E + e \sin E)\right]_{E^0}^{E}$$

These equations represent a complete first-order solution for optimum rendezvous with a nearby elliptic orbit. They contain six arbitrary constants (the λ_i) whose values must be determined so as to satisfy the two-point boundary-value problem of going from the initial values of the six orbital elements a, e, θ_1, θ_2, θ_3, M at time zero to the final values at time t_1. Since Eqs. (30–35) are linear in both the Lagrange multipliers and the changes in the state variables, the two-point boundary-value problem can be solved by straightforward (if somewhat laborious) techniques. Some explicit solutions of the two-point boundary-value problem have been obtained by Gobetz[4] for the special case of transfer between neighboring circular orbits.

If the time for rendezvous is much greater than the period of the elliptic orbit so that rendezvous requires a large number of revolutions, Eqs. (30–37) can be greatly simplified and an explicit solution easily obtained. This is done by neglecting the periodic terms (which remain bounded) in comparison with the dominant secular terms in Eqs. (30–37) to yield Eqs. (38–45):

$$\Delta a = \lambda_1 \left(\frac{a^9}{K^3}\right)^{1/2} 4E + \lambda_6 \left(\frac{a^7}{K^3}\right)^{1/2} 3E^2 \tag{38}$$

$$\Delta e = \lambda_2 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{5}{2} (1-e^2)E \tag{39}$$

$$\Delta\theta_1 = \lambda_3 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{5-4e^2}{2e^2} E - \lambda_6 \left(\frac{a^5}{K^3}\right)^{1/2} \times (1-e^2)^{1/2} \frac{5+8e^2}{2e^2} E \tag{40}$$

$$\Delta\theta_2 = \lambda_4 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{1+4e^2}{2(1-e^2)} E \tag{41}$$

$$\Delta\theta_3 = \lambda_5 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{E}{2} \tag{42}$$

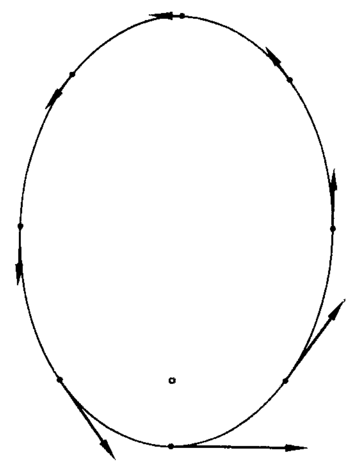

Fig. 1 Optimum thrust program for change in a ($e = 0.707$).

$$M = E - \lambda_1 \left(\frac{a^7}{K^3}\right)^{1/2} 3E^2 - \lambda_3 \left(\frac{a^5}{K^3}\right)^{1/2} (1-e^2)^{1/2} \times \frac{5+8e^2}{2e^2} E - \lambda_6 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{3}{2} E^3 \equiv E - \Delta M \tag{43}$$

$$2J = \lambda_1^2 \left(\frac{a^9}{K^3}\right)^{1/2} 4E + \lambda_1\lambda_6 \left(\frac{a^7}{K^3}\right)^{1/2} 6E^2 + \lambda_2^2 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{5}{2} (1-e^2)E + \lambda_3^2 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{5-4e^2}{2e^2} E - 2\lambda_3\lambda_6 \left(\frac{a^5}{K^3}\right)^{1/2} (1-e^2)^{1/2} \frac{5+8e^2}{2e^2} E + \lambda_4^2 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{1+4e^2}{2(1-e^2)} E + \lambda_5^2 \left(\frac{a^5}{K^3}\right)^{1/2} \frac{E}{2} + \lambda_6^2 \left(\frac{a^5}{K^3}\right)^{1/2} 3E^3 \tag{44}$$

$$t = (a^3/K)^{1/2} E \tag{45}$$

By solving Eqs. (38–43) for the Lagrange multipliers, the payoff may be expressed explicitly in terms of the changes in the elements:

$$\left.\begin{aligned} J &= \frac{K}{2at_1} \left[\frac{\Delta a^2}{4a^2} + \frac{2}{5} \frac{\Delta e^2}{1-e^2} + \frac{2e^2}{5-4e^2} \Delta\theta_1^2 + \frac{2(1-e^2)}{1+4e^2} \Delta\theta_2^2 + 2\Delta\theta_3^2\right] + \frac{2a^2}{3} \frac{(\Delta M - \Delta M^*)^2}{t_1^3} \\ \Delta M^* &\equiv -\frac{3}{4} \left(\frac{K}{a^5}\right)^{1/2} t_1 \Delta a - (1-e^2)^{1/2} \frac{5+8e^2}{5-4e^2} \Delta\theta_1 \end{aligned}\right\} \tag{46}$$

Results

The optimum thrust acceleration programs for changing the five elements that represent the size, shape, and orientation of an elliptic orbit are shown pictorially in Figs. 1–5 for an eccentricity of 0.707. These figures have been calculated from Eqs. (19) by considering all the Lagrange multipliers, but the one corresponding to the element of interest, to be zero. Each of these programs will minimize the fuel required to change its corresponding element in a given time. In general, each of these programs will produce changes in the other elements of the orbit, these changes being given by Eqs. (30–35). However, if only long-time motion is considered, these five programs will not only maximize the rate of change of one element but will produce no change in the other four of the five elements. The programs for maximizing the change of any of five elements produce no secular changes in the other four elements, as may be seen from Eqs. (30–35).

Equations (19) show that there is no secular change in the thrust acceleration programs for changing the five elements

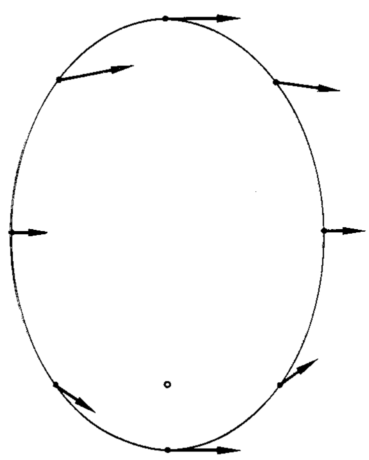

Fig. 2 Optimum thrust program for change in e (e = 0.707).

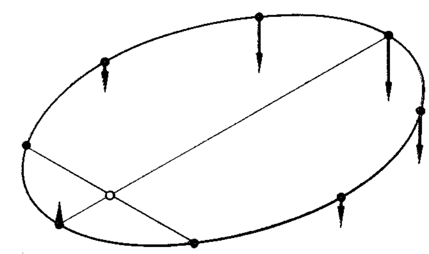

Fig. 4 Optimum thrust program for change in θ_2 (e = 0.707).

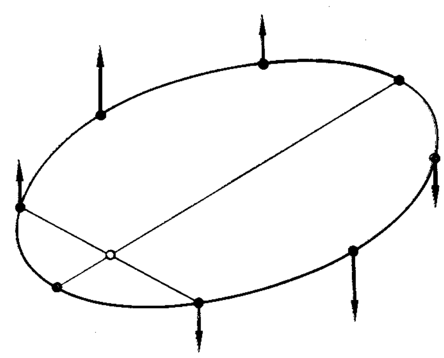

Fig. 5 Optimum thrust program for change in θ_3 (e = 0.707).

of the ellipse. The average thrust acceleration will therefore remain constant from revolution to revolution so that J, for changing these five elements, may be represented as in Eqs. (47) and (48):

$$J = \sum_{i=1}^{5} J_i = \int_0^{t_1} \sum_{i=1}^{5} \frac{A_i^2}{2}\, dt = \sum_{i=1}^{5} \frac{\overline{A_i^2}\, t_1}{2} = \sum_{i=1}^{5} \frac{\bar{A}_i^2 t_1^2}{2t_1} \equiv \frac{\sum_{i=1}^{5} u_i^2}{2t_1} \tag{47}$$

$$u \equiv \left(\sum_{i=1}^{5} u_i^2\right)^{1/2} = (2t_1 J)^{1/2} = \left[\sum_{i=1}^{5} \frac{\Delta x_i^2}{(d\bar{x}_i/du)^2}\right]^{1/2} \tag{48}$$

The quantity u defined in Eqs. (47) and (48) may be interpreted physically as the change in velocity that could be produced by the same propulsion system in the same time and with the same fuel consumption in field-free space. The average rate of change of each of the five elements of the orbit with respect to u, when only that element is changed, is plotted as a function of e in Fig. 6. These average rates of change are the denominators in the third expression in Eq. (48).

For orbit transfer problems where the final position in the orbit is unspecified, only the first five elements of the orbit need be considered. However, for rendezvous problems, the final value of the mean anomaly, the sixth element, must be considered. Equation (46) represents the fuel requirements for changing all six elements over long time periods. The quantity ΔM^* represents the perturbation in the mean anomaly produced by the thrust programs for changing a and $\Delta\theta_1$. The fuel required to change the mean anomaly depends upon the square of the difference between the desired perturbation and this particular perturbation. The thrust program that produces the desired perturbation in mean anomaly is the semimajor axis program of Fig. 1 with an average thrust acceleration that is a linear function of time. The first term in the parentheses following λ_6^0 in Eqs. (19) has a negligible effect on long-time motion compared to the second term which produces the time-varying semimajor axis program.

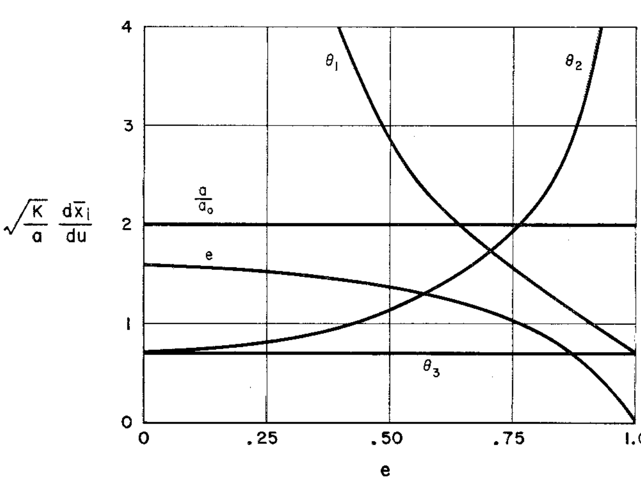

Fig. 6 Maximum average rates of change of orbital elements.

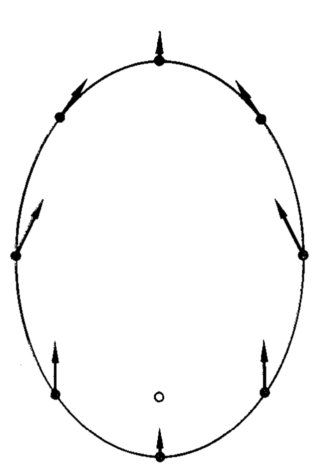

Fig. 3 Optimum thrust program for change in θ_1 (e = 0.707).

Since Eq. (46) holds only in the case of long-time motion, the mean anomaly term of Eq. (46) will generally be negligible compared to the terms in brackets because it is inversely proportional to time cubed. The fuel required for rendezvous will be only slightly greater than the fuel required for orbit transfer for these long-time cases.

The results of this paper may also be applied to low-thrust station keeping. For most applications, it will be sufficient to introduce a secular thrust perturbation that is equal and opposite to the undesired natural secular perturbations. The use of optimum thrust programs should provide appreciable fuel economy relative to the nonoptimum programs considered in Refs. 18 and 19.

This paper has been concerned with minimum-fuel transfer in a specified time from any point on an elliptic orbit to any desired point on a neighboring elliptic orbit. The complete solution is contained in Eqs. (19) for the optimum thrust program and in Eqs. (30–37) for the resulting vehicle motion and fuel consumption. Equations (19) show that there are five thrust programs containing only short-period terms corresponding to five elements of the ellipse, and a sixth thrust program containing both short-period and secular terms corresponding to the mean anomaly. Three of the first five thrust programs produce secular changes only in their corresponding elements, whereas two of these programs also produce secular changes in the mean anomaly. When the transfer time is long so that periodic terms may be neglected, Eqs. (30–37) reduce to Eqs. (38–45) for which the two-point boundary-value is easily solved. However, the two-point boundary-value problem can be solved even in the general case of Eqs. (30–35), since these equations are linear in the Lagrange multipliers and the changes in the elements. The solution of the two-point boundary-value problem completes the solution of the orbit transfer problem considered herein.

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